$\boxed{\text{Theorem (Uniqueness of Taylor Series)}}$ If a power series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n$$

converges to a function $f(z)$ on a disk $D_R(z_0)$, then it is the Taylor series of $f$ about $z_0$.

Proof. We need to show that $a_n = \frac{f^{(n)}(z_0)}{n!}$. Consider $g(z) = \frac{1}{2\pi i} \cdot \frac{1}{(z-z_0)^{n+1}}$ where $n \geq 0$. Let $C$ be a circle centered at $z_0$ w/ radius $r < R$.

$$\frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad \left(\text{Cauchy Int. Formula}\right)$$

$$= \int_C g(z) \sum_{m=0}^{\infty} a_m (z-z_0)^m \, dz$$

$$= \sum_{m=0}^{\infty} a_m \int_C g(z)(z-z_0)^m \, dz \quad \left(\text{Integrating Power Series}\right)$$

$$= \sum_{m=0}^{\infty} \frac{a_m}{2\pi i} \int_C \frac{1}{(z-z_0)^{(n+1)-m}}\, dz$$

$$= a_n \quad \left(\text{by an example}\right).$$

▨

$\boxed{\text{Theorem (Uniqueness of Laurent Series)}}$ If a series

$$\sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} b_n \frac{1}{(z-z_0)^n}$$

converges to a function $f(z)$ on an annulus $R_1 < |z-z_0| < R_2$, then it is the Laurent series for $f$ on that annulus.

Proof. Similar to the proof of uniquess of Taylor series. $\square$

## Multiplication of Power Series

Suppose two power series

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n \quad \text{and} \quad g(z) = \sum_{n=0}^{\infty} b_n (z-z_0)^n$$

converge on a disk $D_R(z_0)$. Then $f$ and $g$ are analytic on that disk and hence so is $f \cdot g$, by the product rule. Hence, $f \cdot g$ has a Taylor series on $D_R(z_0)$:

$$f(z)\, g(z) = \sum_{n=0}^{\infty} c_n \,(z - z_0)^n$$

with coefficients

$$\frac{n!}{k!(n-k)!}$$

$$\begin{aligned}
c_n &= \frac{(fg)^{(n)}(z_0)}{n!} \\
&= \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} f^{(k)}(z_0)\, g^{(n-k)}(z_0) \qquad \text{(Liebniz)} \\
&= \sum_{k=0}^{n} \frac{f^{(k)}(z_0)}{k!} \, \frac{g^{(n-k)}(z_0)}{(n-k)!} \\
&= \sum_{k=0}^{n} a_n b_{n-k} .
\end{aligned}$$

Usually, only the first several terms are required. They can be found by formally multiplying the series like polynomials.

$\boxed{\text{Example}}$ Find the Maclaurin series for

$$f(z) = \frac{\sinh z}{1+z}.$$

The function $\sinh z$ and $\frac{1}{1+z}$ are analytic on the unit disk. We have

$$\sinh z \cdot \frac{1}{1+z} = \left(\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}\right)\left(\sum_{n=0}^{\infty} (-1)^n z^n\right)$$

$$= \left(z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots\right)\left(1 - z + z^2 - \cdots\right)$$

$$\begin{aligned} = \; & z + \frac{z^3}{3!} + \frac{z^5}{5!} \\ & - z^2 - \frac{z^4}{3!} - \frac{z^6}{5!} \\ & + z^3 + \frac{z^5}{3!} + \frac{z^7}{5!} \\ & - z^4 - \frac{z^6}{3!} - \frac{z^8}{5!} + \end{aligned}$$

$$= z - z^2 + \frac{7}{6} z^3 - \frac{7}{6} z^4 + \cdots\cdots$$

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Similarly, if $f(z)$ and $g(z)$ are analytic on a disk $D_R(z_0)$ and $g(z) \neq 0$ on $D_R(z_0)$, then we can write

$$\frac{f(z)}{g(z)} = \sum_{n=0}^{\infty} d_n (z - z_0)^n$$

where

$$d_n = \frac{\left(\frac{f}{g}\right)^{(n)}(z_0)}{n!}.$$

In fact, the coefficients turn out to be the same as those found by dividing the series like polynomials.

Example Find the Laurent series for

$$f(z) = \frac{1}{\sinh z}$$

on the annulus $0 < |z| < \pi$. We have

$$\frac{1}{\sinh z} = \frac{1}{\sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}} = \frac{1}{z + \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots}$$

$$= \frac{1}{z}\left(\frac{1}{1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots}\right)$$

The series $1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots$ is nonzero on the disk $|z| < \pi$, so we can divide.

$$\begin{array}{r|l}
 & 1 - \frac{z^2}{3!} + \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots \\
\hline
1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \cdots & 1 \\
 & -\left(1 + \frac{z^2}{3!} + \frac{z^4}{5!} + \right) \\
\hline
 & -\frac{z^2}{3!} - \frac{z^4}{5!} - \cdots \\
 & -\left(-\frac{z^2}{3!} - \frac{z^4}{(3!)^2} - \quad\right) \\
\hline
 & \left(\frac{1}{(3!)^2} - \frac{1}{5!}\right) z^4 + \cdots
\end{array}$$

Hence,
$$\frac{1}{\sinh z} = \frac{1}{z}\left(1 - \frac{z^2}{3!} + \left(\frac{1}{(3!)} - \frac{1}{5!}\right)z^4 + \dots\right)$$
$$= \frac{1}{z} - \frac{z}{6} + \frac{7}{360}z^3 - \dots$$

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# Chapter 6: Residues and Poles

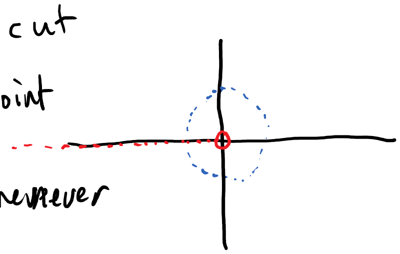
**Definition (Isolated Singularity)** A singular point $z_0$ of a function $f$ is *isolated* if there exists a deleted disk $D_\varepsilon(z_0)\setminus\{z_0\}$ on which $f$ is analytic.

**Examples**

(1) A rational function $F(z) = \frac{P(z)}{Q(z)}$ has only isolated singularities. They are the zeros of $Q(z)$.

(2) The principal branch of the logarithm has a singularity at 0. It is not isolated since any deleted disk about 0 contains points on the branch cut

(3) $f(z) = \frac{1}{\sin\left(\frac{\pi}{z}\right)}$ has a singular point



at $z=0$. Also, it has singular points whenever

$$\sin\frac{\pi}{z} = 0 \iff \frac{\pi}{z} = k\pi \text{ for } k \in \mathbb{Z}$$
$$\iff z = \frac{1}{k} \text{ for } k \in \mathbb{Z}.$$

The singular point $z=0$ is not isolated. Let $D_\varepsilon(0)\setminus\{0\}$ be a deleted neighborhood about 0. Since $\varepsilon > 0$, choose $k \in \mathbb{N}$ such that $\frac{1}{k} < \varepsilon$. Then $z = \frac{1}{k} \in D_\varepsilon(0)\setminus\{0\}$, but $f$ is not analytic at $z = \frac{1}{k}$.

The singularities $z = \frac{1}{k}$ are isolated since $f$ is analytic on the deleted disk $D_{\frac{1}{k(k+1)}}\left(\frac{1}{k}\right)\setminus\left\{\frac{1}{k}\right\}$. //

$\boxed{\text{Definition (Isolated Singularity at } \infty)}$ A function $f(z)$ has an isolated singularity at $\infty$ if there exists $R > 0$ such that $f$ is analytic on the annulus $R < |z| < \infty$.

$\boxed{\text{Definition (Residues)}}$ Let $z_0$ be an isolated singularity of $f$ so that $f$ is analytic on an annulus

$$\begin{cases} 0 < |z - z_0| < R, & z_0 \neq \infty \\ R < |z| < \infty, & z_0 = \infty. \end{cases}$$

When $z_0 \neq \infty$, the residue of $f$ at $z_0$ is the coefficient

$$\operatorname*{Res}_{z=z_0} f(z) \overset{\text{def}}{=} b_1 = \frac{1}{2\pi i}\int_C f(z)\,dz$$

in the Laurent series expansion of $f$. When $z_0 = \infty$, the residue of $f$ at $\infty$ is defined via

$$\operatorname*{Res}_{z=\infty} f(z) \overset{\text{def}}{=} \frac{1}{2\pi i}\int_{C_{R_0}} f(z)\,dz$$

where $C_{R_0}$ is a negatively oriented circle centered at 0 with radius $R_0 > R$. //

## Example

(1) Compute $\int_C \frac{e^z-1}{z^4}\,dz$ where $C$ is the unit circle w/ positive orientation. Since zero is an isolated singularity of $\frac{e^z-1}{z^4}$, and $C$ is a contour about 0, we need only compute $\operatorname{Res}_{z=0} \frac{e^z-1}{z^4}$. The function has a Laurent series on $0<|z|<\infty$. We have

$$\frac{e^z-1}{z^4} = \frac{1}{z^4}\left(\sum_{n=0}^{\infty} \frac{z^n}{n!} - 1\right) = \frac{1}{z^4}\sum_{n=1}^{\infty} \frac{z^n}{n!}$$

$$= \sum_{n=1}^{\infty} \frac{z^{n-4}}{n!}$$

$$= \frac{1}{z^3} + \frac{1}{2z^2} + \frac{1}{6z} + \sum_{n=0}^{\infty} \frac{z^n}{(n+4)!}$$

Hence, $\operatorname{Res}_{z=0} \frac{e^z-1}{z^4} = \frac{1}{6}$ and $\int_C \frac{e^z-1}{z^4}\,dz = 2\pi i \operatorname{Res}_{z=0} \frac{e^z-1}{z^4}$

$$= \frac{\pi i}{3}.$$

(2) Compute $\int_C \cosh\left(\frac{1}{z^2}\right) dz$ where $C$ is the unit circle w/ positive orientation. The function $\cosh \frac{1}{z^2}$ has an isolated singularity at 0 and it is analytic on the annulus $0<|z|<\infty$. We have

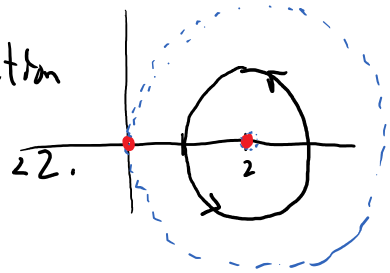
$$\cosh \frac{1}{z^2} = \sum_{n=0}^{\infty} \frac{\left(\frac{1}{z^2}\right)^{2n}}{(2n)!}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^{4n}(2n)!} = 1 + \frac{1}{2z^4} + \frac{1}{24z^8} + \cdots +$$

So $\operatorname{Res}_{z=0} \cosh \frac{1}{z^2} = 0$ and $\int_C \cosh \frac{1}{z^2} = 2\pi i \cdot 0 = 0.$

(3) Compute $\displaystyle\int_C \frac{1}{z(z-2)^5}\,dz$ where $C$ is the circle $|z-2|=1$ w/ positive orientation.

We need to compute $\displaystyle\operatorname*{Res}_{z=2} \frac{1}{z(z-2)^5}$. The function has a Laurent series on the annulus $0<|z-2|<2$. Note that this condition implies $\left|\frac{z-2}{2}\right|<1$.




We have

$$\frac{1}{z(z-2)^5} = \frac{1}{(z-2)^5}\left(\frac{1}{2+(z-2)}\right)$$

$$= \frac{1}{2(z-2)^5}\left(\frac{1}{1+\frac{z-2}{2}}\right)$$

$$= \frac{1}{2(z-2)^5}\sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^n}{2^n} = \sum_{n=0}^{\infty}(-1)^n\frac{(z-2)^{n-5}}{2^{n+1}}$$

The residue occurs when $n=4$. Hence, $\displaystyle\operatorname*{Res}_{z=2}\frac{1}{z(z-2)^5} = \frac{1}{32}$.

Hence,
$$\int_C \frac{1}{z(z-2)^5}\,dz = \frac{\pi i}{16}.$$

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$\boxed{\text{Theorem (Residue Theorem)}}$ Let $C$ be a positively oriented simple closed contour. If $f$ is analytic everywhere on and interior to $C$, except at a finite number of singularities $z_1, \dots, z_n$ lying interior to $C$, then

$$\int f(z)\,dz = 2\pi i\sum_{1}^{n}\operatorname{Res} f(z).$$

Then

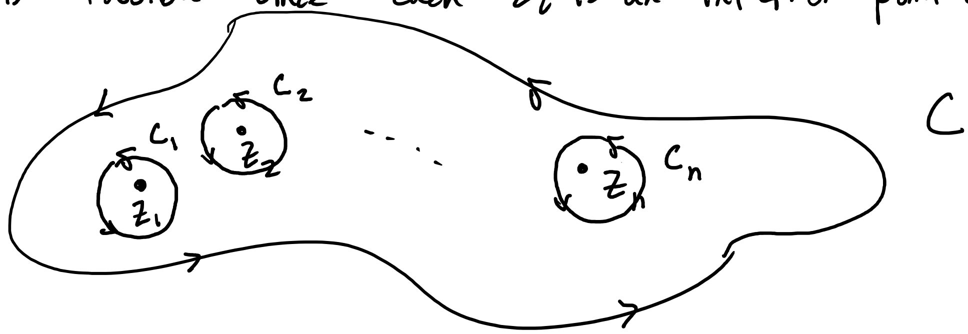
$$\int_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z).$$

Proof. (C.f. Midterm M4) The singularities $z_1, \ldots, z_n$ are isolated since there are finitely many. For each $i = 1, \ldots, n$, let $C_i$ be a positively oriented circle centered at $z_i$ such that

(1) the regions $R_i$ enclosed by $C_i$ are pairwise disjoint.

(2) the regions $R_i$ enclosed by $C_i$ lies interior to $C$.

Note that (2) is possible since each $z_i$ is an interior point of $C$.




Then $f$ is analytic everywhere on $C, C_1, \ldots, C_n$ and at all points that are interior to $C$ but exterior to each $C_i$. By the Generalized Cauchy Goursat theorem,

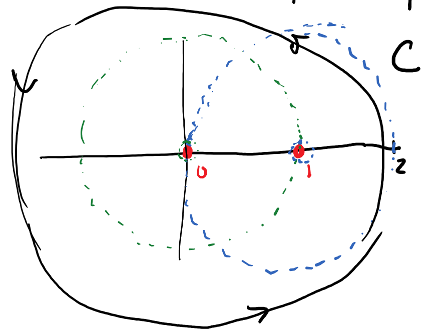
$$\int_C f(z)\,dz = \sum_{i=1}^{n} \int_{C_i} f(z)\,dz$$

$$= \sum_{i=1}^{n} 2\pi i \operatorname*{Res}_{z=z_i} f(z)$$

$$= 2\pi i \sum_{i=1}^{n} \operatorname*{Res}_{z=z_i} f(z)$$

$\boxed{\text{Example}}$ Compute $\displaystyle\int_C \frac{4z-5}{z(z-1)}\,dz$ over the circle $|z| = 2$ with positive orientation. The function has isolated singularities at $z = 0, 1$, both of which lies interior to $C$. So, we apply the residue theorem. The function has

at $z=0,1$, both of which lies interior to $C$. So, we apply the residue theorem. The function has a Laurent series on the annulus $0<|z|<1$. We have




$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z}\left(\frac{1}{z-1}\right)$$

$$= \frac{5-4z}{z}\left(\frac{1}{1-z}\right)$$

$$= \left(\frac{5}{z} - 4\right)\sum_{n=0}^{\infty} z^n \quad = \frac{5}{z}\sum_{n=0}^{\infty} z^n - 4\sum_{n=0}^{\infty} z^n$$

$$= 5\sum_{n=0}^{\infty} z^{n-1} - 4\sum_{n=0}^{\infty} z^n$$

$$= \frac{5}{z} + \sum_{n=0}^{\infty} z^n$$

So $\operatorname*{Res}_{z=0} f(z) = 5.$

To compute $\operatorname*{Res}_{z=1} f(z)$, we seek a Laurent series on the annulus $0<|z-1|<1$. We have

$$\frac{4z-5}{z(z-1)} = \frac{4z-5}{z-1}\left(\frac{1}{1+(z-1)}\right)$$

$$= \frac{4(z-1)-1}{z-1}\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$= \left(4 - \frac{1}{z-1}\right)\sum_{n=0}^{\infty}(-1)^n(z-1)^n$$

$$= 4\sum_{n=0}^{\infty}(-1)^n(z-1)^n - \sum_{n=0}^{\infty}(-1)^n(z-1)^{n-1}$$

$$= 4\sum_{n=0}^{\infty}(-1)^n(z-1)^n - \sum_{n=1}^{\infty}(-1)^n(z-1)^{n-1} - \frac{1}{z-1}$$

Hence, $\operatorname*{Res}_{z=1} f(z) = -1$ and

$$\int_C \frac{4z-5}{z(z-1)}\,dz = 2\pi i(5-1) = 8\pi i.$$

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